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Excitable Structures in Stochastic Bistable Media

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We examine the influence of parametric noise on the spatiotemporal behavior of a bistable medium with activator–inhibitor dynamics. Deterministic front propagation in one dimension is seen to be destabilized by the external noise, resulting in the propagation of solitary pulses through the system. For large enough noise levels, this state becomes unstable via a backfiring mechanism, which eventually leads to a turbulent state.

KEY WORDS: Reaction diffusion system; noise-induced phenomena; activator–inhibitor dynamics; pulse propagation.

Noise has a relevant influence on the dynamics of bistable systems, due to the arousal of interwell transitions that would not exist in purely deterministic conditions. Particularly important, realizations of such an influence occur in systems to which a small-amplitude external signal is applied. In that case, signal amplification can be obtained for an optimal amount of noise, giving rise to phenomena such as stochastic resonance,[1–4] noise-enhanced transmission of information[5] and noise-enhanced phase coherence.[6]

When *spatially extended* media are considered, the interplay between local spatial coupling and noise effects leads to further interesting phenomena.[7] In particular, external noise has been seen to substantially affect front, spiral and wave propagation in autonomous systems,[8–11] to sustain spatiotemporal structures in excitable media[12–14] which might explain noise enhanced propagation of structures in neuroscience,[15] to enhance signal propagation in bistable systems,[16–18] to favor the decay of metastable states in periodically driven systems[19] and to induce domain growth in conserved-order-parameter systems.[20,21]

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In the case of bistable systems, however, most of the spatiotemporal phenomena analysed so far are linked to a very simple scenario, namely the propagation of a front, or trigger wave,[22, 23] connecting the two stable stales of the system. In the present paper, we argue that in a bistable medium with activator-inhibitor dynamics, spatiotemporal parametric noise is able to support non-trivial dynamical structures arising from simple local perturbations of a homogenous state, such as running solitary pulses and turbulent-like states driven by backfiring processes. Traveling pulses are frequently observed in chemical systems,[22] and are considered to be the mechanism of signal propagation in neural systems,[24] but so far they have been usually studied only in the framework of excitable media. Here we show that this behavior can also arise in bistable media in a natural way, as long as noise sources (ubiquitous for instance in neural tissue) exist in the system.

We consider an activator-inhibitor model defined in a one-dimensional space. In dimensionless units the model reads[25]

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{\varepsilon}\, u(1-u)\left(u - \frac{v+b}{a}\right) + \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= [\gamma + \eta(x,t)]\, u - v \end{aligned} \tag{1}$$

where $u(x, t)$ and $v(x, t)$ are the concentrations of the activator and the inhibitor, respectively. Only the activator is allowed to diffuse through the medium. The two species have well-separated time scales, in such a way that the dynamics of the activator is much faster than that of the inhibitor. This is taken into account by the parameter ε, which represents the ratio of decay times of the two species ($\varepsilon = \tau_u/\tau_v$), and that will be considered to be much smaller than unity. Parameters a, b, and γ are chosen in such a way that the system is in a bistable regime. The inhibitor-production coefficient is assumed to fluctuate around a mean value γ, as represented by the spatiotemporal white noise $\eta(x, t)$, which has zero mean and correlation defined by

$$\langle \eta(x,t)\, \eta(x',t) \rangle = 2\sigma^2 \delta(t-t')\, \delta(x-x') \tag{2}$$

The local dynamics of the system in the absence of noise can be easily described in the phase plane (u, v). Figure 1 shows the nullclines $\dot{u} = 0$ and $\dot{v} = 0$ in that plane. Crossings of these nullclines correspond to fixed points of the model. For the parameters chosen here, the system displays two stable fixed points (labeled 0 and 1 in the figure) and an unstable one

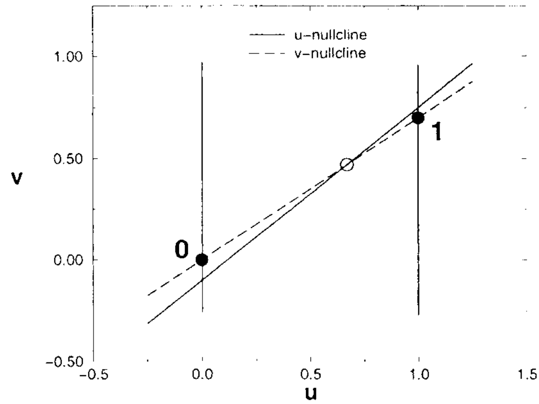

Fig. 1. Nullclines of model (1) in the absence of noise, for $a = 0.85$, $b = 0.1$, and $\gamma = 0.7$. Solid circles denote stable points, and the empty circle indicates an unstable fixed point.

(denoted by an empty circle). Transitions between the two stable fixed points can be induced by an intense enough perturbation (as in simpler one-variable double-well systems).

In a bistable extended medium resting in an homogeneous steady state (0 or 1 in Fig. 1), an initial local perturbation of sufficient intensity is able to induce a local transition towards the other stable steady state, and spatial coupling triggers similar transitions in the neighborhood. This gives rise to a propagating front, as shown in Fig. 2a. In that plot the space-time evolution of model (1) is represented in the absence of noise. The system is initially in state 0 (coded in white in the figure), and a perturbation is applied to the leftmost site. Under these conditions, a transition towards state 1 (coded in black in the figure) is seen to propagate through the system with a fixed velocity. Simulations are performed in a discrete chain of $N = 400$ sites with spacing $\Delta x = 0.25$ and absorbing boundary conditions. The activator equation is integrated with a semi-implicit algorithm proposed by Barkley,[(26)] and the inhibitor equation by means of an explicit Euler method.

We are now interested in analyzing the effect of the parametric noise defined by (2) on the previous simple behavior. We first note that the effect of such a noise on the local dynamics described in Fig. 1 is to introduce fluctuations in the slope of the v-nullcline. In this way, the system might undergo local transitions to the excitable regime, which occurs for large enough values of the slope, for which state 1 collides with the unstable fixed point and becomes unstable itself. It is to be expected that such transitions lead to the destabilization of the front solution and, as we will see in what follows, this is indeed the case.

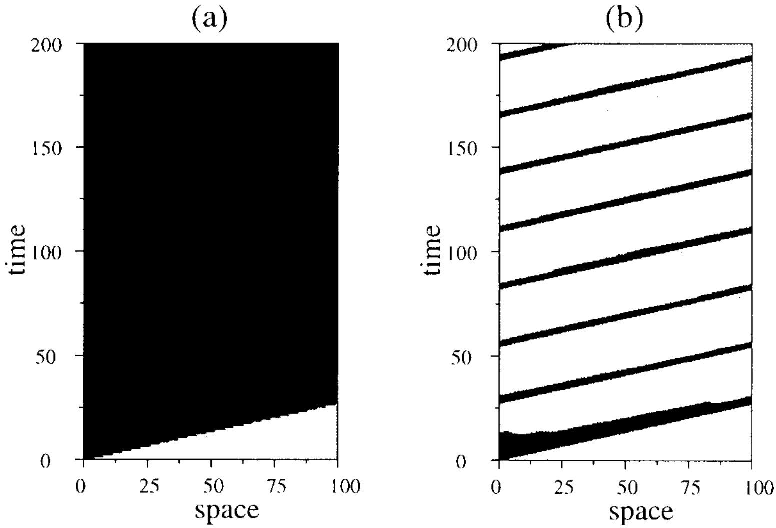

Fig. 2. Spatiotemporal evolution of model (1). The profile of the activator u is plotted (horizontally) at increasing times (vertically). Black coding corresponds to $u = 1$, and white to $u = 0$. Parameters used are those of Fig. 1, plus $\varepsilon = 0.01$. Noise intensity is: (a) $\sigma^2 = 0.0$; (b) $\sigma^2 = 0.0015$.

The result of adding a small amount of parametric noise to the system is shown in Fig. 2b. In this case periodic boundary conditions are used, and as a result a single solitary pulse propagating indefinitely towards the right is obtained.[27] While destabilization of the front solution by noise should not be surprising (fluctuations push locally the system away from state 1), the way in which the medium self-organizes to generate a stable traveling pulse is certainly remarkable. In order to understand why this occurs, we plot in Fig. 3 the superimposed instantaneous profiles of both the activator and inhibitor variables for one of these pulses at a given time instant.

As revealed by Fig. 3, the evolution of the activator is virtually free from fluctuations, and displays only jumps between the two fixed points of the local dynamics. These jumps, rather abrupt due to the small value of ε, delimit the boundaries of the traveling pulse. The mechanisms that originate the front and the rear of the pulse are very different. On one hand, the front of the pulse corresponds to the propagation of state 1 ("excited" state, using the terminology of excitable media) into a noise free region in state 0 ("rest" state), and therefore it is basically a deterministic process. The rear jump, on the other hand, corresponds to the propagation of the rest region into a fluctuating excited region. This propagation is initiated by noise, and the corresponding front adjusts its speed to that of

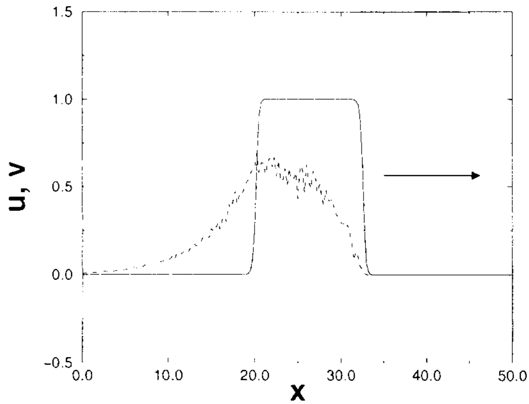

Fig. 3. Spatial profiles of the activator (solid line) and the inhibitor (dashed line) for a traveling pulse. Parameters are those of Fig. 1, plus $\varepsilon = 0.01$ and $\sigma^2 = 2 \times 10^{-4}$.

the first one, similarly to what occurs in excitable media.[23] Therefore, the speed of the noise-initiated pulse is the same as that of the deterministic front. This fact can be observed by bare-eye inspection of Fig. 2, and has also been systematically checked in our simulations.

Even though the running pulses shown in Fig. 2b have been observed to be stable for long times (up to 5000 tune units), their width experiences fluctuations die to the external noise. We have computed the relative dispersion of the pulse width, defined as $\Delta \equiv \sqrt{\langle (w - \langle w \rangle)^2 \rangle} / \langle w \rangle$, where w is the pulse width at half maximum, and the averages have been performed over both time evolution and ensemble realizations. The results are shown in Fig. 4a for increasing noise intensity σ^2 and two values of the relaxation time ratio ε. The Figure also shows (plot (b)) the corresponding number of pulses living in the system at a given time instant, averaged again over time and ensemble realizations. It can be observed (Fig. 4a) that for a rather broad range of small enough noise levels, the pulse-width dispersion remains below 15%, independently of the parameter ε. This behavior roughly corresponds (see Fig. 4b) to the situation in which a single pulse travels through the system. This is the case shown in Fig. 2a.

For larger noise intensities, some realizations exhibit behaviors in which more than one pulse exists at a given time, resulting in an average number of pulses greater than one. Before examining what is the spatiotemporal dynamics in this case, and addressing the issue of its origin, it is instructive to look at the behavior of the absolute width of the pulses for increasing noise levels. This information is plotted in Fig. 5 for the same range of noise intensities as in Fig. 4. It is clear that, as long as the number

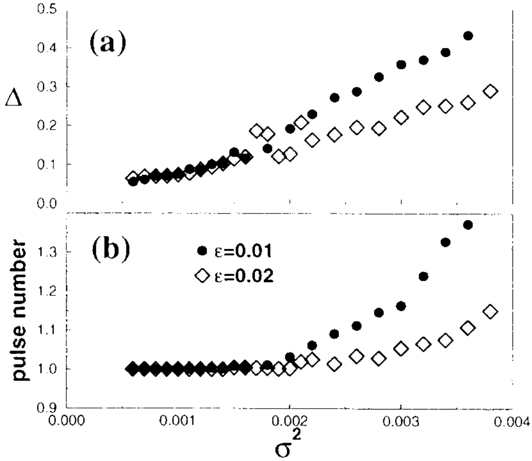

Fig. 4. Relative dispersion of the pulse width (a) and average number of pulses in the system (b) for increasing noise intensity and two different values of the relaxation-time ration ε. Parameters are those of Fig. 1.

of pulses in the system is strictly kept equal to unity, the pulse width increases monotonically with noise intensity. We can understand this behavior by taking into account that a large noise intensity implies large inhibitor fluctuations in the excited (i.e., $u \approx 1$) region. We recall that the front of the pulse is deterministically driven, and only its tail depends on the existence of fluctuations. This tail corresponds to a front propagating

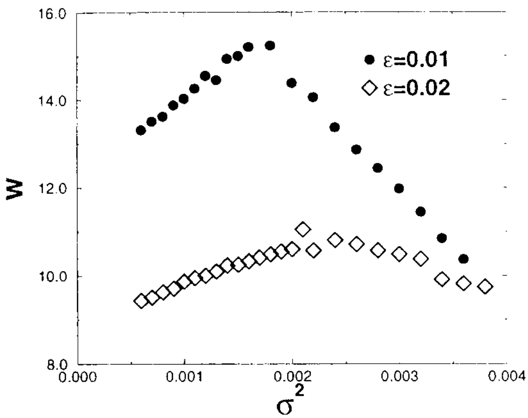

Fig. 5. Pulse width at half maximum for increasing noise intensity. Parameters are those of Fig. 4.

from a resting (i.e., $u \approx 0$) towards a excited region. This propagation, will be hindered by the existence of large local concentrations of inhibitor in the excited region, such as those produced by intense noise in our model. Therefore, it is to be expected that the larger the noise level, the wider the pulses, as it is certainly observed.

The monotonic increase in the pulse width with noise intensity breaks down for sufficiently large noise levels. At this point, the fluctuations are intense enough to induce a transition from the excited to the rest state in the middle of the wide pulse, breaking it in two halves. One half continues to move towards the right at the deterministically prescribed speed, whereas the other half starts to move in the opposite direction, with the same absolute velocity. This phenomenon, known as *backfiring*, is shown in the spatiotemporal plot of Fig. 6a. This regime corresponds to a substantial increase of the pulse-width dispersion Δ (Fig. 4a) and a decrease of the absolute pulse width with increasing noise (Fig. 5). The reason for such a decrease is that now the backfiring mechanism dominates the whole dynamics of the system, breaking the pulse in two before its tail can adjust its propagation velocity to that of the pulse front, as explained earlier in the framework of the single–pulse regime.

The frequency of the backfiring events becomes larger with increasing noise intensity, eventually leading to a state of spatiotemporal turbulence such as the one shown in Fig. 6b. In this figure, we have applied absorbing

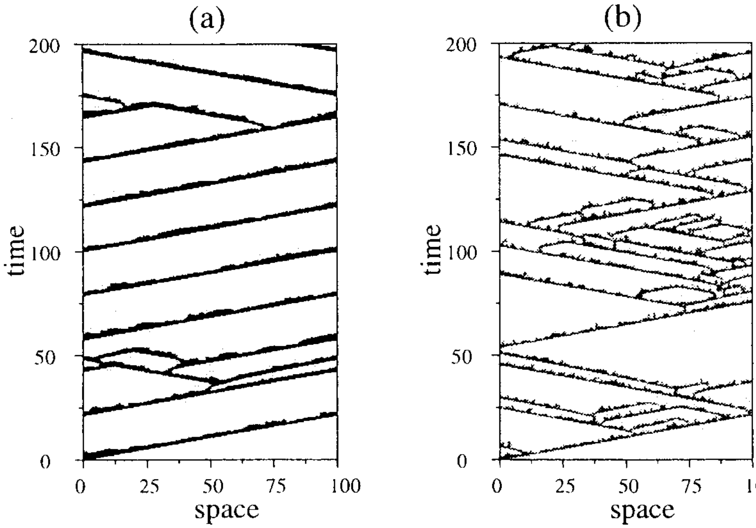

Fig. 6. Spatiotemporal evolution of model (1) for large noise intensities. Parameters are those of Fig. 2. Noise intensities are: (a) $\sigma^2 = 0.005$; (b) $\sigma^2 = 0.02$.

boundary conditions in order to prevent circulating pulses from reducing the complexity of the dynamics through annihilations with other pulses. We should emphasize at this point that backfiring and the associated turbulent behavior reported here are not observed in the deterministic version of simple activator inhibitor models such as the one examined in this paper. In the absence of fluctuations, backfiring requires non-standard dynamics of the inhibitor variable, that, lead to more that one fixed point in the excitable regime.[28, 29]

In conclusion, we have observed that spatiotemporal external noise is able to support excitable-like structures in a bistable medium with simple activator–inhibitor dynamics. For the considered values of the parameter ($\varepsilon \propto 0.01 \rightarrow 0.02$) the dynamics is approximately that of a simple threshold and time scales of the activator and inhibitor are widely separated. Then front propagation is destabilized by noise, and the system self organizes in such a way that stable pulse propagation appears. The pulses are seen to have a very well defined width, with a dispersion no larger that 15%. The width of the pulses increases with noise intensity. For larger levels of noise, the pulses become unstable themselves through backfiring processes that split them in half. This backfiring leads, for intense enough noise, to turbulent behavior.

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